Combinatorial Counting - 3.4 - 3.6 Estimates I

Main question: Which function is growing (significantly) faster?

Harmonic numbers are $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Question: $\lim_{n\to\infty} H_n = \infty$. But how fast it grows? Is it like n or $\sqrt{n}$ or $\log(n)$ or $\log^* n$?

1: Let G_k be numbers $\frac{1}{i}$ such that $\frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}}$. What is $|G_k|$? Find a simple upper and lower bounds on $\sum_{x\in G_k} x$, maybe use a lower or upper bound that works for all terms in the sum.

Solution:

$$G_k = \left\{ \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdot + \frac{1}{2^{k-1}+2^{k-1}-1} \right\}$$

Now it is easy to see that $|G_k| = 2^{k-1}$.

$$\frac{1}{2} = 2^{k-1} \cdot \frac{1}{2^k} < \sum_{x\in G_k} x \leq 2^{k-1} \cdot \frac{1}{2^{k-1}} = 1$$

2: Notice that

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \leq \sum_{k=1}^{\ell} \sum_{x\in G_k} x$$

Find ℓ and use the previous exercise to establish an upper bound on H_n.

Solution: The last term in the sum is $\frac{1}{n}$, which satisfies $\frac{1}{2^\ell} < \frac{1}{n} \leq \frac{1}{2^{\ell-1}}$ for some k. For upper bound, we need Or in other words, $n < 2^\ell$. By taking $\log_2$, we get $\log_2 n < \ell$. This is satisfied by $\ell = \lfloor \log_2 n \rfloor + 1$. Hence

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \leq \sum_{k=1}^{\ell} \sum_{x\in G_k} x \leq \sum_{k=1}^{\ell} 1 = \ell = \lfloor \log_2 n \rfloor + 1.$$

3: Use a similar idea as in the previous exercise to establish a lower bound on H_n.

Solution:

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \geq \sum_{k=1}^{\ell} \sum_{x\in G_k} x \leq \sum_{k=1}^{\ell} \frac{1}{2} = \frac{\ell}{2}.$$

Now we need to find the largest ℓ such that the above holds. Taking $\ell = \lfloor \log_2 n \rfloor$ works. Hence

$$H_n \geq \frac{1}{2} \lfloor \log_2 n \rfloor.$$